UCLA COURSE FIELD NOTES

MATH 33B

Differential Equations

First-order models, linear equations, systems, stability, and power-series solutions

How these notes are written. I wrote each chapter the way I wish the material had been explained the first time: the real idea, the point where I usually get stuck, the simplest mental model that still stays honest, and a worked decision instead of a polished answer appearing from nowhere. Every chapter closes with practice and a fully written solution. If a sentence sounds impressive but does not help me solve or explain something, it does not belong here.

Daniel Mastick

Curriculum map, working notes, practice, and solutions

Course map

Week	Core thread	What should be solid by the end
1	**Modeling and direction fields**	Translate rates, read qualitative behavior, and verify candidate solutions.
2	**Separable equations**	Separate variables, handle equilibria, and apply initial conditions.
3	**First-order linear equations**	Integrating factors and superposition.
4	**Autonomous equations**	Equilibria, phase lines, stability, logistic growth, and long-term behavior.
5	**Second-order homogeneous**	Characteristic roots: distinct, repeated, and complex.
6	**Forced equations**	Undetermined coefficients, resonance, and variation of parameters.
7	**Mechanical models**	Oscillation, damping, forcing, and energy-aware interpretation.
8	**Linear systems**	Eigenvalue solutions and fundamental matrices.
9	**Phase-plane stability**	Nodes, saddles, spirals, centers, and repeated eigenvalues.
10	**Power series**	Ordinary points, recurrence relations, and local analytic solutions.

Scope anchor: https://catalog.registrar.ucla.edu/course/2025/MATH33B. The sequence is aligned to the official catalog description and expanded into a practical ten-week study plan.

How I would use this packet

Treat each numbered section as one chapter. Read the formal explanation, pause at the student interpretation, and see whether the easy version still says the same thing. Rework the example without looking, then cover the solution in the chapter practice box. The cumulative set at the end is deliberately mixed: knowledge becomes durable when I have to choose the tool instead of being told which chapter I am in.

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CHAPTER 1 An equation for a function

Rebuild the chapter from first principles

The claim I need to own is this: The model in An equation for a function controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of an equation for a function. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach an equation for a function on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the course foundations to separable equations. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A differential equation asks for an entire function whose rates satisfy a rule. The order tells how much initial data I need, and checking a proposed solution means substituting both the function and its derivatives.

The easy version

Instead of solving for a number, solve for a curve whose slope or acceleration follows the rule.

An ordinary differential equation relates an unknown function to its derivatives. A solution is not a number; it is a function satisfying the relation on an interval.

The initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0$$

selects a trajectory through (t_0, y_0). Before solving, a direction field shows where solutions rise, fall, flatten, or approach equilibria.

The move

After finding a formula, substitute it into the differential equation and the initial condition. Solving algebraically can introduce or lose branches; verification restores the contract.

Solved chapter practice

Practice. Verify that $y = Ce^{2x} + 3/2$ solves $y' - 2y = -3$.
Solution. $y' = 2Ce^{2x}$. Then $y' - 2y = 2Ce^{2x} - 2(Ce^{2x} + 3/2) = -3$, so every constant C gives a solution.

Mastery practice A. Reconstruct an equation for a function from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a an equation for a function problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 2 Separable equations

Rebuild the chapter from first principles

The claim I need to own is this: The model in Separable equations controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of separable equations. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach separable equations on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from an equation for a function to first-order linear equations. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

Separation works when all y-dependence can sit with dy and all x-dependence with dx. I check equilibrium solutions before dividing, because dividing by a factor of y can silently erase them.

The easy version

Put the variables on opposite sides, integrate both sides, then use the initial condition.

If $y' = g(t)h(y)$, rearrange:

$$\frac{dy}{h(y)} = g(t)dt.$$

Integrate both sides, then use the initial condition.

Worked example

Solve $y' = ty$, $y(0) = 2$.
Solution. For $y \neq 0$, $dy/y = t\,dt$, so $\ln|y| = t^2/2 + C$. Thus $y = Ce^{t^2/2}$. The initial condition gives

$$y = 2e^{t^2/2}.$$

Common miss

Dividing by $h(y)$ can erase equilibrium solutions where $h(y) = 0$. Identify equilibria before dividing and add them back if needed.

Solved chapter practice

Practice. Solve $y' = xy$, $y(0) = 4$.
Solution. For $y \neq 0$, $dy/y = x\,dx$. Thus $\ln|y| = x^2/2 + C$, so $y = Ce^{x^2/2}$. The initial condition gives $C = 4$, hence $y = 4e^{x^2/2}$.

Mastery practice A. Reconstruct separable equations from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a separable equations problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent

check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 3 First-order linear equations

Rebuild the chapter from first principles

The claim I need to own is this: The model in First-order linear equations controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of first-order linear equations. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach first-order linear equations on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from separable equations to autonomous equations and stability. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

The integrating factor is designed to turn the left side into one product derivative. I put the equation in standard form first; otherwise the factor is built from the wrong coefficient.

The easy version

Multiply by a specially chosen function so two awkward terms collapse into one derivative.

Standard form is

$$y' + p(t)y = q(t).$$

The integrating factor $\mu(t) = e^{\int p(t)dt}$ turns the left side into $(\mu y)'$:

$$\boxed{(\mu y)' = \mu q.}$$

Worked example

Solve $y' + 2y = e^{-t}$.
Solution. $\mu = e^{2t}$, so $(e^{2t}y)' = e^{t}$. Integrating gives $e^{2t}y = e^{t} + C$, hence

$$y = e^{-t} + Ce^{-2t}.$$

The first term is a particular response; the second is the decaying homogeneous memory of the initial condition.

Solved chapter practice

Practice. Solve $y' + 2y = e^{-x}$.
Solution. The integrating factor is $\mu = e^{\int 2dx} = e^{2x}$. Then $(e^{2x}y)' = e^{x}$. Integrating gives $e^{2x}y = e^{x} + C$, so $y = e^{-x} + Ce^{-2x}$.

Mastery practice A. Reconstruct first-order linear equations from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a first-order linear equations problem but never states a model or checks the result. Write the shortest useful review of that solution.

Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 4 Autonomous equations and stability

How I actually think about it

A phase line lets me study long-term behavior without solving explicitly. Equilibria are zeros of $f(y)$ in $y' = f(y)$, and the sign between them tells which way solutions move.

The easy version

Mark the resting levels, then draw arrows showing whether the state rises or falls between them.

For $y' = f(y)$, equilibria satisfy $f(y) = 0$. A phase line uses the sign of f between equilibria to show flow.

The logistic model

$$y' = ry\left(1 - \frac{y}{K}\right)$$

has equilibria 0 and K. For $r > 0$, 0 is unstable and K is asymptotically stable for positive initial data.

The idea

An explicit solution answers where the system is at time t. A phase portrait answers what every nearby initial condition will do. The second view is often more durable.

Rebuild the chapter from first principles

The claim I need to own is this: An explicit solution answers where the system is at time t. A phase portrait answers what every nearby initial condition will do. The second view is often more durable. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of autonomous equations and stability. List the known quantities, the unknown, the units or types, and any hidden constraint.

2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
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- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach autonomous equations and stability on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from first-order linear equations to second-order homogeneous equations. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

Solved chapter practice

Practice. Classify equilibria of $y' = y(2-y)$.
Solution. Equilibria are $y = 0, 2$. For $y < 0$, $y' < 0$; for $0 < y < 2$, $y' > 0$; for $y > 2$, $y' < 0$. Arrows move away from 0, so it is unstable, and toward 2, so it is stable.

Mastery practice A. Reconstruct autonomous equations and stability from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains

why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a autonomous equations and stability problem but never states a model or checks the result. Write the shortest useful review of that solution.

Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 5 Second-order homogeneous equations

Rebuild the chapter from first principles

The claim I need to own is this: The model in Second-order homogeneous equations controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of second-order homogeneous equations. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I

do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach second-order homogeneous equations on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from autonomous equations and stability to forced equations and resonance. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: the characteristic polynomial converts constant-coefficient differentiation into algebra. Real roots give exponentials, repeated roots need an extra x, and complex roots encode oscillation through sine and cosine.

The easy version

Guess exponential behavior, solve a polynomial for the allowed growth rates, and combine the resulting modes.

For

$$ay'' + by' + cy = 0,$$

try $y = e^{rt}$. The characteristic equation is $ar^2 + br + c = 0$.

- Distinct real roots r_1, r_2: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- Repeated root r: $y = (C_1 + C_2 t)e^{rt}$.
- Complex roots $\alpha \pm i\beta$: $y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$.

Two initial conditions are needed because the solution space has dimension two.

Solved chapter practice

Practice. Solve $y'' - 4y' + 4y = 0$.
Solution. The characteristic equation is $(r-2)^2 = 0$, a repeated root $r = 2$. Therefore $y = (C_1 + C_2 x)e^{2x}$.

Mastery practice A. Reconstruct second-order homogeneous equations from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a second-order homogeneous equations problem but never states a model or checks the result. Write the shortest useful review of that solution.

Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 6 Forced equations and resonance

Rebuild the chapter from first principles

The claim I need to own is this: The model in Forced equations and resonance controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of forced equations and resonance. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach forced equations and resonance on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from second-order homogeneous equations to mechanical models. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: the homogeneous solution is the system's natural behavior; a particular solution is its response to forcing. Resonance happens when forcing matches a natural mode, so the usual guess overlaps the homogeneous solution and must be multiplied by x.

The easy version

The total motion is what the system does on its own plus what the outside input makes it do.

For $L[y] = g(t)$, the general solution is $y = y_h + y_p$. Undetermined coefficients guesses a particular form when g is built from polynomials, exponentials, sines, or cosines.

Worked example

Solve $y'' + y = \cos t$.
Solution. The forcing duplicates a homogeneous solution, so the ordinary guess $A\cos t + B\sin t$ fails. Multiply by t and try $y_p = At\sin t$. Substitution gives $A = 1/2$:

$$y = C_1\cos t + C_2\sin t + \frac{1}{2}t\sin t.$$

The growing amplitude is resonance.

Variation of parameters handles more general forcing. For a fundamental pair y_1, y_2 and Wronskian $W = y_1y_2' - y_1'y_2$ in $y'' + py' + qy = g$,

$$y_p = -y_1\int\frac{y_2 g}{W}\,dt + y_2\int\frac{y_1 g}{W}\,dt.$$

Solved chapter practice

Practice. Find a particular solution of $y'' + y = \sin x$.
Solution. Because $\sin x$ is already a homogeneous mode, try $y_p = x(A\cos x + B\sin x)$. Substitution gives $A = -1/2$, $B = 0$, so one particular solution is $y_p = -\frac{1}{2}x\cos x$.

Mastery practice A. Reconstruct forced equations and resonance from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a forced equations and resonance problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 7 Mechanical models

Rebuild the chapter from first principles

The claim I need to own is this: The model in Mechanical models controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of mechanical models. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.

- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach mechanical models on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from forced equations and resonance to linear systems. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: mass-spring equations are Newton's second law written with restoring and damping forces. The coefficients have units and physical meanings, so the discriminant predicts whether the motion oscillates or returns without crossing equilibrium.

The easy version

Mass resists acceleration, the spring pulls home, and damping removes motion.

The mass-spring-damper equation is

$$my'' + cy' + ky = F(t).$$

No damping gives sustained oscillation; underdamping gives decaying oscillation; critical damping returns fastest without oscillating; overdamping returns more slowly without oscillating.

Sanity check

Root real parts control growth or decay; imaginary parts control oscillation. This interpretation transfers directly to first-order systems.

Solved chapter practice

Practice. Classify $y'' + 6y' + 9y = 0$ and solve for $y(0) = 1$, $y'(0) = 0$.
Solution. The root is repeated: $(r + 3)^2 = 0$, so the system is critically damped and $y = (C_1 + C_2t)e^{-3t}$. From $y(0) = 1$, $C_1 = 1$. Differentiating and using $y'(0) = 0$ gives $C_2 = 3$, so $y = (1 + 3t)e^{-3t}$.

Mastery practice A. Reconstruct mechanical models from a blank page. Include the central

definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a mechanical models problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 8 Linear systems

Rebuild the chapter from first principles

The claim I need to own is this: The model in Linear systems controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of linear systems. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.

- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach linear systems on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from mechanical models to phase-plane classification. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: a first-order system packages interacting rates into matrix form. Eigenvectors give invariant directions and eigenvalues determine growth, decay, or oscillation along them.

The easy version

Several changing quantities feed one another; the matrix records who affects whom.

For $\mathbf{x}' = A\mathbf{x}$, an eigenpair $A\mathbf{v} = \lambda\mathbf{v}$ gives solution $e^{\lambda t}\mathbf{v}$. With enough independent eigenvectors,

$$\mathbf{x}(t) = \sum_i c_i e^{\lambda_i t}\mathbf{v}_i.$$

Complex eigenvalues combine into real oscillatory solutions. Repeated defective eigenvalues require generalized eigenvectors and terms such as $te^{\lambda t}$.

For a nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$, superposition again separates homogeneous dynamics from a particular response.

Solved chapter practice

Practice. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = (3, 4)$.

Solution. The components decouple: $x_1' = x_1$, $x_2' = -2x_2$. Hence $x_1 = 3e^t$ and $x_2 = 4e^{-2t}$, so $\mathbf{x} = (3e^t, 4e^{-2t})$.

Mastery practice A. Reconstruct linear systems from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.

Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a linear systems problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 9 Phase-plane classification

Rebuild the chapter from first principles

The claim I need to own is this: The model in Phase-plane classification controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of phase-plane classification. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach phase-plane classification on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from linear systems to power-series solutions. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

The trace, determinant, and eigenvalues describe local geometry near an equilibrium. I still look at eigenvectors because they show the actual directions of approach or escape, especially for saddles.

The easy version

Eigenvalues say grow, decay, or rotate; eigenvectors draw the tracks those behaviors follow.

For a 2×2 linear system:

- real eigenvalues with opposite signs: saddle, unstable;
- both negative: stable node; both positive: unstable node;
- complex with negative real part: stable spiral; positive: unstable spiral;
- purely imaginary, in the ideal linear case: center.

Trace-determinant analysis uses characteristic polynomial $\lambda^2 - \tau\lambda + \Delta$, where $\tau = \operatorname{tr} A$ and $\Delta = \det A$.

Solved chapter practice

Practice. Classify the origin for $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{x}$.

Solution. The characteristic polynomial is $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$. Both eigenvalues are real, distinct, and negative, so the origin is an asymptotically stable node.

Mastery practice A. Reconstruct phase-plane classification from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a phase-plane classification problem but never states a model or checks the result. Write the shortest useful review of that solution.

Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 10 Power-series solutions

Rebuild the chapter from first principles

The claim I need to own is this: The model in Power-series solutions controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of power-series solutions. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach power-series solutions on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from phase-plane classification to cumulative practice. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: a power series turns the differential equation into a recurrence for coefficients. The index shift is the delicate part: every sum must use the same power of x before coefficients can be compared.

The easy version

Assume the solution is an infinite polynomial, substitute it, and let the equation tell you each coefficient from earlier ones.

Near an ordinary point, assume

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n, \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitute, align powers, and equate coefficients. The differential equation becomes a recurrence for a_n; initial values set a_0 and a_1.

Worked example

For $y'' + y = 0$, coefficient matching gives

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.$$

Even and odd coefficients split into the cosine and sine series.

Solved chapter practice

Practice. Find the coefficient recurrence for $y'' + y = 0$ with $y = \sum_{n=0}^{\infty} a_n x^n$.
Solution. $y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$. Matching coefficients in $y'' + y = 0$ gives $(n+2)(n+1)a_{n+2} + a_n = 0$, so $a_{n+2} = -a_n/[(n+2)(n+1)]$.

Mastery practice A. Reconstruct power-series solutions from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a power-series solutions problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 11 Cumulative practice

Rebuild the chapter from first principles

The claim I need to own is this: The model in Cumulative practice controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of cumulative practice. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach cumulative practice on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from power-series solutions to written solutions. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Solve $y' = 2ty^2$, $y(0) = 1$ and state its maximal interval.
2. Solve $y' - y = t$, $y(0) = 0$.
3. Classify equilibria of $y' = y(2 - y)(y - 5)$.
4. Solve $y'' - 5y' + 6y = 0$, $y(0) = 1$, $y'(0) = 0$.
5. Find a particular solution of $y'' + 4y = 3\sin t$.
6. Classify the origin for $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}$.
7. Solve $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$.
8. Find the recurrence for a power-series solution of $y'' - xy = 0$.

CHAPTER 12 Written solutions

Rebuild the chapter from first principles

The claim I need to own is this: The model in Written solutions controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of written solutions. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach written solutions on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from cumulative practice to the cumulative course model. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Separable blow-up

$y^{-2}dy = 2tdt$, so $-1/y = t^2 + C$. The initial condition gives $C = -1$, hence

$$y = \frac{1}{1-t^2}.$$

The maximal interval containing 0 is $(-1, 1)$.

2. Linear equation

$y' - y = t$ has integrating factor e^{-t}:

$$(e^{-t}y)' = te^{-t}.$$

Integration gives $y = Ce^t - t - 1$. From $y(0) = 0$, $C = 1$, so $y = e^t - t - 1$.

3. Phase line

Equilibria are $0, 2, 5$. Sign analysis gives flow $+$ on $(-\infty, 0)$, $-$ on $(0, 2)$, $+$ on $(2, 5)$, and $-$ on $(5, \infty)$. Thus 0 and 5 are stable; 2 is unstable.

4. Second order IVP

Roots are $2, 3$, so $y = C_1e^{2t} + C_2e^{3t}$. Conditions give $C_1 + C_2 = 1$ and $2C_1 + 3C_2 = 0$, hence $C_1 = 3$, $C_2 = -2$.

5. Forced response

Try $y_p = A\sin t + B\cos t$. Then $y_p'' + 4y_p = 3A\sin t + 3B\cos t$. Therefore $A = 1$, $B = 0$, so $y_p = \sin t$.

6. Saddle

Eigenvalues are 1 and -2, with opposite signs. The origin is a saddle and unstable.

7. Defective system

$A = 2I + N$ with $N^2 = 0$, so $e^{At} = e^{2t}(I + tN)$. Thus

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

8. Series recurrence

Substitution gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0.$$

Thus $a_2 = 0$ and, for $n \geq 1$,

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}.$$

Closing frame

Differential equations model systems by writing the rule of change instead of the final state. The reusable skill is to separate structure from forcing, then read stability from the homogeneous dynamics. A formula matters; the shape of the solution family matters more.