UCLA COURSE FIELD NOTES

MATH 32B

Calculus of Several Variables

Multiple integrals, coordinate changes, vector fields, and the major integral theorems

How these notes are written. I wrote each chapter the way I wish the material had been explained the first time: the real idea, the point where I usually get stuck, the simplest mental model that still stays honest, and a worked decision instead of a polished answer appearing from nowhere. Every chapter closes with practice and a fully written solution. If a sentence sounds impressive but does not help me solve or explain something, it does not belong here.

Daniel Mastick

Curriculum map, working notes, practice, and solutions

Course map

Week	Core thread	What should be solid by the end
1	**Double integrals**	Iterated integration, Fubini's theorem, and regions of Type I and II.
2	**Polar coordinates**	Circular geometry, the factor r, and region design.
3	**Triple integrals**	Solids, mass, moments, and average value.
4	**Cylindrical and spherical**	Coordinate choice driven by symmetry.
5	**Change of variables**	Jacobians and transformed regions.
6	**Line integrals**	Scalar work along curves and vector-field circulation.
7	**Conservative fields**	Potential functions and path independence.
8	**Green's theorem**	Trade planar boundary circulation for interior curl.
9	**Surface integrals**	Parametrized surfaces, flux, and orientation.
10	**Stokes and divergence**	Boundary-interior duality in three dimensions.

Scope anchor: https://catalog.registrar.ucla.edu/course/2025/MATH32B. The sequence is aligned to the official catalog description and expanded into a practical ten-week study plan.

How I would use this packet

Treat each numbered section as one chapter. Read the formal explanation, pause at the student interpretation, and see whether the easy version still says the same thing. Rework the example without looking, then cover the solution in the chapter practice box. The cumulative set at the end is deliberately mixed: knowledge becomes durable when I have to choose the tool instead of being told which chapter I am in.

Contents

CHAPTER 1 Double integrals and region literacy

Rebuild the chapter from first principles

The claim I need to own is this: The model in Double integrals and region literacy controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of double integrals and region literacy. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach double integrals and region literacy on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the course foundations to polar coordinates. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

The integrand is only half the problem; the region is the other half. I sketch first, because wrong bounds can produce flawless algebra for the wrong shape.

The easy version

Describe the floor plan, then add the tiny columns standing on it.

The double integral accumulates over area:

$$\iint_R f(x,y)dA.$$

For continuous f on a rectangular region, Fubini lets us compute by iterated integrals. For a Type I region $a \le x \le b$, $g_1(x) \le y \le g_2(x)$,

$$\iint_R f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

The move

Sketch the region before writing bounds. Inner bounds trace a slice; outer bounds sweep those slices across the region. If one order requires splitting and the other does not, switch.

Worked example

Over the triangle $0 \le y \le x \le 1$,

$$\iint_R (x+y)dA = \int_0^1 \int_0^x (x+y)dydx = \int_0^1 \frac{3}{2}x^2 dx = \frac{1}{2}.$$

Area is $\iint_R 1dA$; average value is $\frac{1}{\text{area}(R)} \iint_R f\,dA$.

Solved chapter practice

Practice. Evaluate $\iint_R (x+y)\,dA$ on $0 \le x \le 1$, $0 \le y \le 2$.
Solution. $\int_0^1 \int_0^2 (x+y)\,dy\,dx = \int_0^1 (2x+2)dx = [x^2+2x]_0^1 = 3.$

Mastery practice A. Reconstruct double integrals and region literacy from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains

why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a double integrals and region literacy problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 2 Polar coordinates

Rebuild the chapter from first principles

The claim I need to own is this: The model in Polar coordinates controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of polar coordinates. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
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- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach polar coordinates on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from double integrals and region literacy to triple integrals and physical moments. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: polar coordinates are useful when the region or integrand speaks in circles. The extra r in $dA = r\,dr\,d\theta$ is not optional; it accounts for wedges spreading as radius grows.

The easy version

Use radius and angle for round regions, and remember that outer wedges are wider than inner wedges.

$x = r\cos\theta$, $y = r\sin\theta$, and

$$\boxed{dA = r\,dr\,d\theta.}$$

The factor r is local area scaling: a fixed angular slice is wider farther from the origin.

Worked example

Integrate $e^{-(x^2+y^2)}$ over the disk $x^2+y^2 \le a^2$:

$$\int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta = \pi(1-e^{-a^2}).$$

Common miss

The polar integrand is not complete until dA becomes $r\,dr\,d\theta$. Forgetting the Jacobian factor changes the geometry, not just a coefficient.

Solved chapter practice

Practice. Find the area inside $r = 2\cos\theta$.
Solution. The circle is traced for $-\pi/2 \le \theta \le \pi/2$. Area is $\frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\cos\theta)^2 d\theta = 2\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = \pi$.

Mastery practice A. Reconstruct polar coordinates from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.

Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a polar coordinates problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 3 Triple integrals and physical moments

Rebuild the chapter from first principles

The claim I need to own is this: The model in Triple integrals and physical moments controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of triple integrals and physical moments. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.

- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach triple integrals and physical moments on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from polar coordinates to change of variables and jacobians. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A triple integral adds tiny volume elements. Density changes what each tiny piece contributes, and moments multiply by distance before adding. Units tell me whether I am computing volume, mass, or rotational inertia.

The easy version

Fill the solid with tiny boxes; weight each box by density and any required distance factor.

$$\iiint_E f(x, y, z)dV$$

accumulates through volume. With density ρ, mass is $m = \iiint_E \rho\, dV$ and center of mass is

$$\bar{x} = \frac{1}{m}\iiint_E x\rho\, dV, \quad \bar{y} = \frac{1}{m}\iiint_E y\rho\, dV, \quad \bar{z} = \frac{1}{m}\iiint_E z\rho\, dV.$$

Symmetry can settle a coordinate before any integration. If E and ρ are symmetric under $x \mapsto -x$, then $\bar{x} = 0$.

Cylindrical coordinates use $dV = r\, dz\, dr\, d\theta$. Spherical coordinates

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

use

$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.}$$

Sanity check

Use the coordinates that make both the region and the integrand simpler. A sphere in rectangular coordinates is possible; it is rarely kind.

Solved chapter practice

Practice. Find the mass of the unit cube with density $\rho = x + y + z$.
Solution. $M = \int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dz\, dy\, dx$. Each variable contributes $1/2$ after the other integrations, so $M = 3/2$.

Mastery practice A. Reconstruct triple integrals and physical moments from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a triple integrals and physical moments problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 4 Change of variables and Jacobians

Rebuild the chapter from first principles

The claim I need to own is this: The model in Change of variables and Jacobians controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of change of variables and jacobians. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
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- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach change of variables and jacobians on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from triple integrals and physical moments to line integrals. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A change of variables reshapes the coordinate grid. The Jacobian is the local area or volume scale factor, so I treat it as geometry rather than a determinant inserted at the end.

The easy version

New coordinates stretch tiny squares; the Jacobian tells how much their area changes.

For $x = x(u, v)$ and $y = y(u, v)$,

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv, \qquad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

The determinant measures signed local area scaling; absolute value gives area.

Worked example

Let $u = x + y$, $v = x - y$. Then $x = (u + v)/2$, $y = (u - v)/2$ and

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{1}{2}.$$

A diamond bounded by lines $x + y = \pm a$ and $x - y = \pm b$ becomes the rectangle $|u| \le a$, $|v| \le b$.

Solved chapter practice

Practice. Let $u = x + y$, $v = x - y$. Express dA in terms of $du\,dv$.
Solution. Solving gives $x = (u+v)/2$, $y = (u-v)/2$. Thus $|\partial(x, y)/\partial(u, v)| = |(-1/4) - (1/4)| = 1/2$, so $dA = \frac{1}{2} du\,dv$.

Mastery practice A. Reconstruct change of variables and jacobians from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a change of variables and jacobians problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 5 Line integrals

Rebuild the chapter from first principles

The claim I need to own is this: The model in Line integrals controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of line integrals. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
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- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach line integrals on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from change of variables and jacobians to conservative fields. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A line integral accumulates along a path. For scalar fields I am adding density times arc length; for vector fields I am measuring how much the field pushes along the direction of travel.

The easy version

Walk along a wire: add scalar weight per step, or add the field's forward push per step.

For a scalar field along $\mathbf{r}(t)$,

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt.$$

For a vector field $\mathbf{F}$, work or circulation is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt.$$

Reversing orientation leaves scalar arc-length integrals unchanged but negates vector line integrals.

Solved chapter practice

Practice. Compute $\int_C (y, x) \cdot d\mathbf{r}$ along $\mathbf{r}(t) = (t, t^2)$, $0 \le t \le 1$.
Solution. $\mathbf{F}(\mathbf{r}(t)) = (t^2, t)$ and $\mathbf{r}' = (1, 2t)$. The dot product is $t^2 + 2t^2 = 3t^2$. Integrating gives $\int_0^1 3t^2 dt = 1$.

Mastery practice A. Reconstruct line integrals from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a line integrals problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 6 Conservative fields

Rebuild the chapter from first principles

The claim I need to own is this: The model in Conservative fields controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of conservative fields. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct

procedure under the wrong assumptions is still a wrong answer.

- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach conservative fields on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from line integrals to green's theorem. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A conservative field stores work in a potential, so path details disappear and only endpoints remain. Before hunting for a potential, I check the domain; holes can break the usual curl-zero conclusion.

The easy version

If the field is a gradient, traveling between the same endpoints costs the same no matter which route you take.

If $\mathbf{F} = \nabla\phi$, then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A).}$$

Work depends only on endpoints. On a simply connected domain, continuous partials plus zero curl imply conservativeness.

For $\mathbf{F} = \langle P, Q\rangle$, find ϕ by integrating P with respect to x, adding an unknown function of y, and matching $\phi_y = Q$.

Common miss

Zero curl on a domain with a hole may not imply a global potential. Domain topology is part of the theorem's contract.

Solved chapter practice

Practice. Find a potential for $\mathbf{F} = (2xy + 1, x^2 + 2y)$ and compute work from $(0, 0)$ to $(1, 2)$.
Solution. Integrate F_1 in x: $\phi = x^2y + x + g(y)$. Matching $\phi_y = x^2 + g' = x^2 + 2y$ gives $g = y^2$. Work is $\phi(1, 2) - \phi(0, 0) = 2 + 1 + 4 = 7$.

Mastery practice A. Reconstruct conservative fields from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a conservative fields problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 7 Green's theorem

How I actually think about it

Green's theorem exchanges circulation around a boundary for curl spread across the interior. Orientation matters: positive orientation keeps the region on my left as I walk around it.

The easy version

Instead of walking the whole fence, total the tiny rotations inside the yard.

For a positively oriented simple closed curve C bounding D,

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)dA.}$$

This trades boundary circulation for interior scalar curl. The flux form is

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA.$$

The idea

Green, Stokes, and divergence are one architectural pattern: integrate a derivative over an interior, or integrate the original field over the oriented boundary.

Rebuild the chapter from first principles

The claim I need to own is this: Green, Stokes, and divergence are one architectural pattern: integrate a derivative over an interior, or integrate the original field over the oriented boundary. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of green's theorem. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

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- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach green's theorem on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from conservative fields to parametric surfaces and flux. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

Solved chapter practice

Practice. Use Green's theorem to compute $\oint_C(-y\,dx + x\,dy)$ around the unit circle counterclockwise.
Solution. Here $P = -y$, $Q = x$, so $Q_x - P_y = 1 - (-1) = 2$. The integral is $\iint_D 2\,dA = 2\pi$.

Mastery practice A. Reconstruct green's theorem from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a green's theorem problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 8 Parametric surfaces and flux

Rebuild the chapter from first principles

The claim I need to own is this: The model in Parametric surfaces and flux controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of parametric surfaces and flux. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.

- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach parametric surfaces and flux on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from green's theorem to stokes and divergence theorems. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: a parametrization turns a flat parameter grid into a surface. The cross product of tangent vectors gives both orientation and the local area scale; its direction matters for flux.

The easy version

Lay a coordinate net over the surface; two tangent directions make a tiny parallelogram whose cross product is the area arrow.

A surface $\mathbf{r}(u, v)$ has tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. Its area element is

$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\| du\, dv.$$

Flux of $\mathbf{F}$ through an oriented surface is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du\, dv,$$

with cross-product order chosen for the requested orientation.

For a graph $z = g(x, y)$ oriented upward, a convenient vector area element is

$$\langle -g_x, -g_y, 1\rangle dA.$$

Solved chapter practice

Practice. Parametrize $z = x + y$ over $0 \le x, y \le 1$ and find its upward area element.
Solution. Use $\mathbf{r}(x, y) = (x, y, x + y)$. Then $\mathbf{r}_x = (1, 0, 1)$, $\mathbf{r}_y = (0, 1, 1)$, and $\mathbf{r}_x \times \mathbf{r}_y = (-1, -1, 1)$, already upward. Thus $d\mathbf{S} = (-1, -1, 1)dxdy$ and $dS = \sqrt{3}dxdy$.

Mastery practice A. Reconstruct parametric surfaces and flux from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a parametric surfaces and flux problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 9 Stokes and divergence theorems

Rebuild the chapter from first principles

The claim I need to own is this: The model in Stokes and divergence theorems controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of stokes and divergence theorems. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach stokes and divergence theorems on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from parametric surfaces and flux to cumulative practice. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

Stokes generalizes Green: boundary circulation equals surface curl. The divergence theorem compares outward flux through a closed surface with sources inside. I choose the theorem that replaces the harder geometry with the easier one.

The easy version

Stokes trades an edge for a surface; divergence trades a closed shell for its interior.

Stokes:

$$\boxed{\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.}$$

Boundary orientation follows the right-hand rule.

Divergence theorem for a closed surface ∂E oriented outward:

$$\boxed{\iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV.}$$

The move

Before computing, choose the cheaper side. Stokes lets us replace a difficult spanning surface with any easier surface sharing the same boundary. Divergence replaces flux through many closed faces with one volume integral.

Solved chapter practice

Practice. Find the outward flux of $\mathbf{F} = (x, y, z)$ through the sphere $x^2 + y^2 + z^2 = R^2$.
Solution. $\nabla \cdot \mathbf{F} = 3$. By the divergence theorem, flux is $\iiint_B 3\,dV = 3(4\pi R^3/3) = 4\pi R^3$.

Mastery practice A. Reconstruct stokes and divergence theorems from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a stokes and divergence theorems problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 10 Cumulative practice

Rebuild the chapter from first principles

The claim I need to own is this: The model in Cumulative practice controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of cumulative practice. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.

- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach cumulative practice on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from stokes and divergence theorems to written solutions. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Reverse the order of $\int_0^1 \int_x^1 f(x,y)dydx$.
2. Evaluate $\iint_D (x^2+y^2)dA$ over $x^2+y^2 \leq 4$.
3. Find the volume inside the sphere $x^2+y^2+z^2 \leq 9$ and above the cone $\phi = \pi/3$.
4. Use $u = x+y$, $v = x-y$ to find the area of $|x+y| \leq 2$, $|x-y| \leq 1$.
5. Compute $\int_C \langle y, x\rangle \cdot d\mathbf{r}$ from $(0,0)$ to $(2,3)$.
6. Use Green's theorem to compute $\oint_C (-y\,dx + x\,dy)$ for the counterclockwise circle $x^2+y^2=4$.
7. Find upward flux of $\mathbf{F} = \langle 0,0,z\rangle$ through $z = 4-x^2-y^2$, $z \geq 0$.
8. Use divergence theorem for outward flux of $\mathbf{F} = \langle x,y,z\rangle$ through the sphere of radius a.

CHAPTER 11 Written solutions

Rebuild the chapter from first principles

The claim I need to own is this: The model in Written solutions controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of written solutions. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach written solutions on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from cumulative practice to the cumulative course model. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Reverse order

The region is $0 \le x \le 1$, $x \le y \le 1$, equivalently $0 \le y \le 1$, $0 \le x \le y$. Thus $\int_0^1 \int_0^y f(x,y)dxdy$.

2. Polar integral

$$\int_0^{2\pi}\int_0^4 r^2 r\,dr\,d\theta = 2\pi\left[\frac{r^4}{4}\right]_0^2 = 8\pi.$$

3. Spherical cap sector

Above the cone means $0 \le \phi \le \pi/3$. Hence

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 9\pi.$$

4. Transformed area

The uv rectangle has area $4 \cdot 2 = 8$. Since $|\partial(x, y)/\partial(u, v)| = 1/2$, the original area is 4.

5. Potential function

$\mathbf{F} = \nabla(xy)$. Therefore the integral is $xy|_{(0,0)}^{(2,3)} = 6$.

6. Green's theorem

$Q_x - P_y = 1 - (-1) = 2$. Disk area is 4π, so circulation is 8π.

7. Graph flux

On the surface, $z = 4 - r^2$ and only the vertical component contributes:

$$\iint_D (4 - r^2)dA = \int_0^{2\pi}\int_0^2 (4 - r^2)r dr d\theta = 8\pi.$$

8. Divergence theorem

$\nabla \cdot \mathbf{F} = 3$. The sphere volume is $4\pi a^3/3$, so flux is $4\pi a^3$.

Closing frame

32B is less about evaluating longer integrals than about choosing representations. Bounds encode geometry. Jacobians encode local scaling. The three major theorems let boundaries and interiors stand in for each other. The clean solution usually comes from seeing that architecture before touching the algebra.