UCLA COURSE FIELD NOTES

MATH 31B

Integration and Infinite Series

Techniques, applications, convergence, power series, and approximation

How these notes are written. I wrote each chapter the way I wish the material had been explained the first time: the real idea, the point where I usually get stuck, the simplest mental model that still stays honest, and a worked decision instead of a polished answer appearing from nowhere. Every chapter closes with practice and a fully written solution. If a sentence sounds impressive but does not help me solve or explain something, it does not belong here.

Daniel Mastick

Curriculum map, working notes, practice, and solutions

Course map

Week	Core thread	What should be solid by the end
1	**Integral strategy**	FTC review, substitution, symmetry, and choosing a useful form.
2	**Integration by parts**	Reverse the product rule and manage repeated reductions.
3	**Trigonometric methods**	Trig integrals, identities, and trigonometric substitution.
4	**Rational and improper integrals**	Partial fractions, infinite intervals, and singular endpoints.
5	**Geometric applications**	Area, solids of revolution, arc length, and average value.
6	**Sequences**	Limits, monotonicity, boundedness, and recursive behavior.
7	**Series**	Geometric and telescoping series; divergence, integral, and comparison tests.
8	**Absolute versus conditional**	Ratio, root, and alternating-series tests; error bounds.
9	**Power series**	Radius and interval of convergence; calculus with series.
10	**Taylor models**	Taylor polynomials, remainder, and cumulative method selection.

Scope anchor: https://catalog.registrar.ucla.edu/course/2025/MATH31B. The sequence is aligned to the official catalog description and expanded into a practical ten-week study plan.

How I would use this packet

Treat each numbered section as one chapter. Read the formal explanation, pause at the student interpretation, and see whether the easy version still says the same thing. Rework the example without looking, then cover the solution in the chapter practice box. The cumulative set at the end is deliberately mixed: knowledge becomes durable when I have to choose the tool instead of being told which chapter I am in.

Contents

CHAPTER 1 Integration is pattern recognition with bookkeeping

Rebuild the chapter from first principles

The claim I need to own is this: The model in Integration is pattern recognition with bookkeeping controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of integration is pattern recognition with bookkeeping. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach integration is pattern recognition with bookkeeping on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the course foundations to integration by parts. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

Antidifferentiation became manageable when I stopped asking "Which formula is this?" and started asking "Which derivative rule would have produced this integrand?" Substitution is the reverse chain rule, including the boring but essential bounds and differential.

The easy version

Look for an inside expression and its derivative sitting nearby; rename the inside and simplify.

The first move is not to integrate. It is to read the integrand: Is there an inner function beside its derivative? A product whose derivative simplifies? A rational function? A hidden trigonometric identity?

The move

Try, in order: algebraic simplification; substitution; integration by parts; trigonometric identities or substitution; partial fractions. Afterward, differentiate the antiderivative. That last step is the fastest reliable audit.

Substitution reverses the chain rule. If $u = g(x)$, then $du = g'(x)dx$:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

For definite integrals, either change the bounds or return to x before evaluating - never mix the two systems.

Worked example

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\ln 2.$$

Solved chapter practice

Practice. Evaluate $\int_0^1 \frac{2x}{1+x^2}\,dx$.
Solution. Let $u = 1 + x^2$, so $du = 2x\,dx$. The bounds become $u = 1$ and $u = 2$. Therefore $\int_1^2 du/u = \ln 2$.

Mastery practice A. Reconstruct integration is pattern recognition with bookkeeping from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a integration is pattern recognition with bookkeeping problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 2 Integration by parts

Rebuild the chapter from first principles

The claim I need to own is this: The model in Integration by parts controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of integration by parts. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach integration by parts on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from integration is pattern recognition with bookkeeping to trigonometric integrals and substitutions. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

Integration by parts is the reverse product rule. My choice of u should get simpler when differentiated, while dv should be something I can actually integrate. I write the boundary term before the remaining integral so signs do not drift.

The easy version

Move a derivative from the awkward factor onto the factor that gets simpler.

Reverse the product rule:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Choose u to become simpler when differentiated and dv to be easy to integrate. Logarithms and inverse trigonometric functions almost always want to be u.

Worked example

Evaluate $\int xe^x\,dx$.
Solution. Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$:

$$\int xe^x dx = xe^x - \int e^x dx = e^x(x-1) + C.$$

Repeated products of a polynomial with e^{ax}, $\sin bx$, or $\cos bx$ can be handled by repeated parts. For $\int e^x \cos x\,dx$, parts twice returns the original integral; collect it algebraically rather than looping forever.

Common miss

Integration by parts does not guarantee a simpler remaining integral. If the second integral looks worse and there is no return path, reconsider the split.

Solved chapter practice

Practice. Evaluate $\int x \ln x \, dx$ for $x > 0$.
Solution. Choose $u = \ln x$, $dv = x\,dx$. Then $du = dx/x$ and $v = x^2/2$. Hence $\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$.

Mastery practice A. Reconstruct integration by parts from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a integration by parts problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 3 Trigonometric integrals and substitutions

Rebuild the chapter from first principles

The claim I need to own is this: The model in Trigonometric integrals and substitutions controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of trigonometric integrals and substitutions. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach trigonometric integrals and substitutions on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from integration by parts to partial fractions and improper integrals. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: the identities are not random decorations. They reshape powers so that one factor becomes a derivative and the rest becomes a single trigonometric function. For radicals, trigonometric substitution encodes a Pythagorean identity directly into the variable.

The easy version

Rewrite the triangle hidden in the radical or the odd power hidden in the product.

For powers of sine and cosine:

- If the sine power is odd, save one $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$.
- If the cosine power is odd, save one $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$.
- If both are even, use half-angle identities.

Three radicals suggest three substitutions:

$$\sqrt{a^2 - x^2} : x = a\sin\theta, \qquad \sqrt{a^2 + x^2} : x = a\tan\theta,$$
$$\sqrt{x^2 - a^2} : x = a\sec\theta.$$

Worked example

Evaluate $\int \sqrt{9-x^2}\,dx$.
Solution. Let $x = 3\sin\theta$, $dx = 3\cos\theta d\theta$. On the principal range, $\sqrt{9-x^2} = 3\cos\theta$:

$$9\int \cos^2\theta d\theta = \frac{9}{2}(\theta + \sin\theta\cos\theta) + C.$$

Returning to x gives

$$\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin(x/3) + C.$$

Solved chapter practice

Practice. Evaluate $\int \sin^3 x \cos^2 x\,dx$.
Solution. Save one sine: $\sin^3 x = (1-\cos^2 x)\sin x$. Let $u = \cos x$, $du = -\sin x\,dx$. The integral is $-\int(1-u^2)u^2 du = -u^3/3 + u^5/5 + C$, or $-\cos^3 x/3 + \cos^5 x/5 + C$.

Mastery practice A. Reconstruct trigonometric integrals and substitutions from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a trigonometric integrals and substitutions problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 4 Partial fractions and improper integrals

Rebuild the chapter from first principles

The claim I need to own is this: The model in Partial fractions and improper integrals controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of partial fractions and improper integrals. List the known quantities, the unknown, the units or types, and any hidden constraint.

2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach partial fractions and improper integrals on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from trigonometric integrals and substitutions to applications of the integral. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

Partial fractions turns one complicated rational expression into several simple ones whose antiderivatives I recognize. Improper integrals require a limit because infinity and vertical asymptotes are not endpoints I can substitute into.

The easy version

Break the fraction into simpler fractions, and replace every dangerous endpoint with a limit.

For a proper rational function P/Q, factor Q over the reals and decompose. Linear factors receive constants; repeated factors receive every power; irreducible quadratics receive linear numerators.

Worked example

$$\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2},$$

so

$$\int \frac{5x+1}{(x-1)(x+2)}dx = 2\ln|x-1| + 3\ln|x+2| + C.$$

An improper integral is defined through a limit. For example,

$$\int_1^\infty \frac{1}{x^p}dx$$

converges exactly when $p > 1$. Near zero, $\int_0^1 x^{-p}dx$ converges exactly when $p < 1$. Same expression, different seam.

Common miss

Never evaluate across a vertical asymptote as though one antiderivative spans it. Split at every singularity and require every resulting limit to converge.

Solved chapter practice

Practice. Determine whether $\int_1^\infty x^{-3/2}\,dx$ converges and find its value.
Solution. $\int_1^b x^{-3/2}dx = [-2x^{-1/2}]_1^b = 2 - 2/\sqrt{b}$. Taking $b \to \infty$ gives 2, so the integral converges.

Mastery practice A. Reconstruct partial fractions and improper integrals from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a partial fractions and improper integrals problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 5 Applications of the integral

Rebuild the chapter from first principles

The claim I need to own is this: The model in Applications of the integral controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of applications of the integral. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach applications of the integral on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from partial fractions and improper integrals to sequences and the idea of convergence. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: every application is the same accumulation idea with a different tiny piece. The real question is what one slice contributes: area, volume, work, mass, or force. Once the differential piece has correct units, the integral usually follows.

The easy version

Describe one thin slice, write its contribution, then add all slices with an integral.

Area between curves is $\int(\text{top} - \text{bottom})dx$ or $\int(\text{right} - \text{left})dy$. The geometry decides the variable.

For volumes:

$$V = \pi \int_a^b (R^2 - r^2)dx \quad \text{(washers)},$$

$$V = 2\pi \int_a^b (\text{radius})(\text{height})dx \quad \text{(shells)}.$$

Arc length of $y = f(x)$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx,$$

and average value is $f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx$.

Sanity check

Area has squared units, volume cubed units, and average value the same units as the function. A radius measured from the wrong axis is the most common shell-method failure.

Solved chapter practice

Practice. A spring requires 12 N to hold it 0.3 m beyond equilibrium. Find the work to stretch it from 0 to 0.5 m.
Solution. Hooke's law gives $12 = k(0.3)$, so $k = 40$ N/m. Work is $W = \int_0^{0.5} 40x\, dx = 20x^2\big|_0^{0.5} = 5$ J.

Mastery practice A. Reconstruct applications of the integral from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a applications of the integral problem but never states a model or checks the result. Write the shortest useful review of that

solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 6 Sequences and the idea of convergence

Rebuild the chapter from first principles

The claim I need to own is this: The model in Sequences and the idea of convergence controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of sequences and the idea of convergence. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach sequences and the idea of convergence on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from applications of the integral to series: convergence is about the tail. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A sequence converges when its tail can be trapped near one number; the first few terms do not decide anything. I separate boundedness from convergence because a bounded sequence may still oscillate forever.

The easy version

Ignore the beginning and ask where the terms settle as the index gets very large.

A sequence (a_n) converges to L if its tail can be forced arbitrarily close to L. Initial terms do not decide convergence.

Useful facts:

- Rational functions of n are controlled by highest powers.
- Exponentials dominate powers: $n^k/a^n \to 0$ for $a > 1$.
- A monotone bounded sequence converges.
- Recursive sequences often reveal their possible limit by solving $L = f(L)$, but that equation does not prove convergence.

Worked example

If $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + 3/a_n)$, a positive limit must satisfy

$$L = \frac{1}{2}(L + 3/L),$$

so $L = \sqrt{3}$. A full proof also shows positivity, a lower bound by $\sqrt{3}$, and monotone decrease once above that bound.

Solved chapter practice

Practice. Find $\lim_{n\to\infty} \frac{3n^2-n}{2n^2+5}$ and justify the simplification.

Solution. Divide numerator and denominator by n^2: $\frac{3-1/n}{2+5/n^2}$. Since $1/n \to 0$ and $1/n^2 \to 0$, the limit is $3/2$.

Mastery practice A. Reconstruct sequences and the idea of convergence from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a sequences and the idea of convergence problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 7 Series: convergence is about the tail

Rebuild the chapter from first principles

The claim I need to own is this: The model in Series: convergence is about the tail controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of series: convergence is about the tail. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.

- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach series: convergence is about the tail on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from sequences and the idea of convergence to power series and taylor models. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A series is a sequence of partial sums. The individual terms going to zero is necessary but not sufficient; harmonic terms shrink to zero and still accumulate without bound. I choose a test by structure, not by whichever test I remember first.

The easy version

Keep a running total. Convergence means that total settles, not merely that each new deposit gets small.

A series $\sum a_n$ converges when its partial sums $S_N = \sum_{n=1}^{N} a_n$ converge. Necessarily $a_n \to 0$, but that alone is not enough; the harmonic series diverges.

Structural tests

- Geometric: $\sum ar^n$ converges for $|r| < 1$.
- Telescoping: write partial sums and expose cancellation.
- Integral test: for positive decreasing f with $f(n) = a_n$, compare to $\int f$.
- Comparison: bound by a known benchmark.
- Limit comparison: if $a_n/b_n \to c$ with $0 < c < \infty$, both behave alike.

The move

Read the dominant behavior. Rational terms usually compare to a p-series. Factorials and exponentials invite the ratio test. An nth power invites the root test. Alternating signs invite absolute convergence first, then the alternating test.

The alternating-series test requires decreasing magnitudes tending to zero. Its error satisfies $|R_N| \le b_{N+1}$.

Solved chapter practice

Practice. Classify $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ as absolutely convergent, conditionally convergent, or divergent.
Solution. The alternating-series test applies because $1/n$ decreases to 0, so the series converges. The absolute series $\sum 1/n$ is harmonic and diverges. Therefore the original series is conditionally convergent.

Mastery practice A. Reconstruct series: convergence is about the tail from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a series: convergence is about the tail problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 8 Power series and Taylor models

Rebuild the chapter from first principles

The claim I need to own is this: The model in Power series and Taylor models controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of power series and taylor models. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.

3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach power series and taylor models on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from series: convergence is about the tail to cumulative practice. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

My shorthand for this chapter is: my shorthand for this chapter is: a power series is an infinite polynomial whose legal inputs are controlled by a radius of convergence. Taylor polynomials are local models: more terms usually improve accuracy near the center, but the remainder is what makes the approximation honest.

The easy version

Build a polynomial from a function's derivatives at one center, and keep track of how far from that center it remains trustworthy.

A power series centered at a has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

It converges inside a radius R, diverges outside, and requires separate endpoint tests. Within the interval of convergence, it can be differentiated and integrated term by term.

The geometric series is the source code:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

Substitution, differentiation, and integration generate many others.

Taylor's formula around a is

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

when the series actually converges to f. The degree-N remainder can be bounded by

$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!}$$

when $|f^{(N+1)}| \leq M$ between a and x.

Core Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Solved chapter practice

Practice. Find the degree-3 Maclaurin polynomial for e^{2x} and approximate $e^{0.2}$.
Solution. Since $e^{2x} = 1 + 2x + (2x)^2/2! + (2x)^3/3! + \cdots$, $P_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$. At $x = 0.1$, $P_3 = 1 + 0.2 + 0.02 + 0.001333 = 1.221333$, close to $e^{0.2}$.

Mastery practice A. Reconstruct power series and taylor models from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a power series and taylor models problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 9 Cumulative practice

Rebuild the chapter from first principles

The claim I need to own is this: The model in Cumulative practice controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of cumulative practice. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach cumulative practice on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from power series and taylor models to written solutions. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Evaluate $\int x^2 \ln x \, dx$.
2. Evaluate $\int \sin^3 x \cos^2 x \, dx$.
3. Evaluate $\int \frac{3x+7}{x^2+x-2} \, dx$.
4. Determine whether $\int_2^\infty \frac{1}{x(\ln x)^2} dx$ converges; evaluate it if it does.
5. Find the volume obtained by revolving the region under $y = x^2$, $0 \le x \le 1$, about the y-axis.
6. Determine $\lim_{n\to\infty}(1 + 3/n)^n$.
7. Determine convergence of $\sum_{n=2}^\infty \frac{1}{n \ln n}$.
8. Classify $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{\sqrt{n}}$ as absolute, conditional, or divergent.
9. Find the radius and interval of convergence of $\sum_{n=1}^\infty \frac{(x-2)^n}{n3^n}$.
10. Use a degree-4 Maclaurin polynomial to approximate $e^{-0.2}$ and bound the error.

CHAPTER 10 Written solutions

Rebuild the chapter from first principles

The claim I need to own is this: The model in Written solutions controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of written solutions. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach written solutions on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from cumulative practice to the cumulative course model. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Parts

Take $u = \ln x$, $dv = x^2 dx$. Then

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

2. Trigonometric integral

Save one sine: $\sin^3 x = (1 - \cos^2 x) \sin x$. With $u = \cos x$,

$$-\int (u^2 - u^4) du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

3. Partial fractions

$x^2 + x - 2 = (x - 1)(x + 2)$ and

$$\frac{3x + 7}{(x - 1)(x + 2)} = \frac{10/3}{x - 1} - \frac{1/3}{x + 2}.$$

Thus the integral is $\frac{10}{3} \ln |x - 1| - \frac{1}{3} \ln |x + 2| + C$.

4. Improper integral

Let $u = \ln x$:

$$\int_2^b \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln b} u^{-2} du = \frac{1}{\ln 2} - \frac{1}{\ln b}.$$

The limit is $1/\ln 2$, so it converges.

5. Shells

Radius is x, height is x^2:

$$V = 2\pi \int_0^1 x^3 dx = \frac{\pi}{2}.$$

6. Exponential limit

The standard limit $(1 + a/n)^n \to e^a$ gives e^3.

7. Integral test

$f(x) = 1/(x \ln x)$ is positive and decreasing for $x \geq 2$, while

$$\int_2^b \frac{dx}{x \ln x} = \ln(\ln b) - \ln(\ln 2) \to \infty.$$

The series diverges.

8. Conditional convergence

The alternating-series test applies because $1/\sqrt{n}$ decreases to zero. The absolute series is a p-series with $p = 1/2$ and diverges. Convergence is conditional.

9. Power series

The ratio test gives $|x-2|/3 < 1$, so $R = 3$. At $x = 5$ the series is $\sum 1/n$, divergent. At $x = -1$ it is $\sum(-1)^n/n$, convergent. Interval: $[-1, 5)$.

10. Taylor approximation

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

At $x = -0.2$, $P_4(-0.2) \approx 0.8187333$. Since $|e^t| \leq 1$ on $[-0.2, 0]$,

$$|R_4| \leq \frac{0.2^5}{5!} < 2.67 \times 10^{-6}.$$

Closing frame

The course has two halves that are really one idea. Integration asks whether a continuous accumulation can be represented and computed. Series ask whether an infinite discrete accumulation settles to a finite value. In both cases, the method is a controlled limit plus enough structure to make the limit usable.