UCLA COURSE FIELD NOTES

MATH 31A

Differential and Integral Calculus

Limits, derivatives, applications, and the first clean look at accumulation

How these notes are written. I wrote each chapter the way I wish the material had been explained the first time: the real idea, the point where I usually get stuck, the simplest mental model that still stays honest, and a worked decision instead of a polished answer appearing from nowhere. Every chapter closes with practice and a fully written solution. If a sentence sounds impressive but does not help me solve or explain something, it does not belong here.

Daniel Mastick

Curriculum map, working notes, practice, and solutions

Course map

Week	Core thread	What should be solid by the end
1	**Functions and models**	Read formulas, graphs, units, compositions, and inverses as different views of the same object.
2	**Limits and continuity**	Compute limits algebraically and explain what continuity does and does not guarantee.
3	**The derivative**	Move between difference quotients, tangent slopes, rates of change, and derivative notation.
4	**Derivative rules**	Use product, quotient, chain, implicit, and inverse-function rules without losing structure.
5	**Transcendental functions**	Differentiate exponential, logarithmic, and trigonometric models and interpret their rates.
6	**Local approximation**	Use linearization and differentials, including honest error and unit checks.
7	**Global information**	Apply Rolle's theorem and the Mean Value Theorem; read monotonicity and concavity.
8	**Optimization and related rates**	Translate a moving or constrained story into one-variable calculus.
9	**Accumulation**	Build the definite integral from sums and use both parts of the Fundamental Theorem.
10	**Synthesis**	Choose methods on mixed problems and explain why the answer is plausible.

Scope anchor: https://catalog.registrar.ucla.edu/course/2025/MATH31A. The sequence is aligned to the official catalog description and expanded into a practical ten-week study plan.

How I would use this packet

Treat each numbered section as one chapter. Read the formal explanation, pause at the student interpretation, and see whether the easy version still says the same thing. Rework the example without looking, then cover the solution in the chapter practice box. The cumulative set at the end is deliberately mixed: knowledge becomes durable when I have to choose the tool instead of being told which chapter I am in.

Contents

CHAPTER 1 Functions are machines with constraints

How I actually think about it

I used to treat a function as whatever formula was printed after the equals sign. The more useful view is that the formula, allowed inputs, outputs, graph, and units are one object. If I ignore the domain, I am studying a different function.

The easy version

A function is a machine, but the machine also comes with a sign telling us what may go in and what kind of thing comes out.

The idea

A function is not just a formula. It is a rule, a domain, and a codomain. Most early calculus mistakes happen because the formula gets all the attention and the constraints disappear.

Rebuild the chapter from first principles

The claim I need to own is this: A function is not just a formula. It is a rule, a domain, and a codomain. Most early calculus mistakes happen because the formula gets all the attention and the constraints disappear. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of functions are machines with constraints. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct

procedure under the wrong assumptions is still a wrong answer.

- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach functions are machines with constraints on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the course foundations to limits: what the function is approaching. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

If $f(x) = \sqrt{4 - x^2}$, the domain is $[-2, 2]$. Its graph is the upper semicircle of radius 2. Those two facts are doing real work: they tell us where a derivative can exist, what an inverse could look like, and why a later optimization problem has endpoints.

Three views should stay connected:

- **Formula:** good for exact computation.
- **Graph:** good for sign, trend, and geometry.
- **Units:** good for catching nonsense. If $s(t)$ is meters, then $s'(t)$ is meters per second.

Composition and inverse

Composition is a pipeline: $(f \circ g)(x) = f(g(x))$. The output of g has to be legal input for f. An inverse reverses a one-to-one function: $f^{-1}(f(x)) = x$ on the relevant domain.

Worked example

Let $f(x) = x^2 + 1$ with domain $x \geq 0$. Then $f^{-1}(y) = \sqrt{y - 1}$ for $y \geq 1$. The restriction $x \geq 0$ is not decoration; without it, f is not one-to-one.

Checkpoint

For $h(x) = \log(3 - x^2)$, state the domain. Then describe, without calculus, whether the graph is even, odd, or neither.
Answer: $-\sqrt{3} < x < \sqrt{3}$; the graph is even.

Solved chapter practice

Practice. For $f(x) = \sqrt{x-1}$ and $g(x) = 1/(x-2)$, find the domain of $f \circ g$.
Solution. We need $g(x)$ to exist, so $x \neq 2$, and we need $g(x) - 1 \geq 0$. Thus $1/(x-2) \geq 1$. Writing $\frac{3-x}{x-2} \geq 0$ and using a sign chart gives $2 < x \leq 3$. Therefore $\mathrm{dom}(f \circ g) = (2, 3]$.

Mastery practice A. Reconstruct functions are machines with constraints from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a functions are machines with constraints problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 2 Limits: what the function is approaching

Rebuild the chapter from first principles

The claim I need to own is this: The model in Limits: what the function is approaching controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of limits: what the function is approaching. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach limits: what the function is approaching on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from functions are machines with constraints to the derivative: a local rate. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A limit is a question about nearby behavior, not a request to plug in the point no matter what. Substitution is only the first diagnostic. When it gives $0/0$, the expression is telling me that its visible form is hiding a simpler local pattern.

The easy version

Walk close to the input without standing on it and watch where the output is heading.

Limit. We write $\lim_{x\to a} f(x) = L$ when values of $f(x)$ can be forced arbitrarily close to L by taking x sufficiently close to a, with $x \neq a$.

The point is local behavior. The value $f(a)$ may be L, something else, or undefined. For computation, I use this order:

1. Substitute. If the result is an ordinary number, stop.
2. If the result is $0/0$, simplify the structure: factor, rationalize, combine fractions, or use a known limit.
3. Check left and right if signs, absolute values, or piecewise definitions matter.

Worked example

Compute $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$.
Solution. For $x \neq 3$,

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3.$$

Therefore the limit is 6. We did not claim the original expression equals $x+3$ at $x=3$; we only used equality nearby.

Two foundational trigonometric limits are

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.}$$

Angles must be in radians. That requirement later explains why the clean derivative of $\sin x$ is $\cos x$.

Continuity

f is continuous at a when all three pieces line up:

$$f(a) \text{ exists}, \qquad \lim_{x\to a} f(x) \text{ exists}, \qquad \lim_{x\to a} f(x) = f(a).$$

The Intermediate Value Theorem says a continuous function on $[a,b]$ takes every value between $f(a)$ and $f(b)$. It guarantees existence, not uniqueness and not a formula.

Common miss

An infinite limit is not a real-valued limit. Writing $\lim_{x\to a} f(x) = \infty$ describes unbounded growth; it does not say the limit exists as a real number.

Checkpoint

Choose c so $f(x) = \frac{x^2-1}{x-1}$ for $x \neq 1$ and $f(1) = c$ is continuous.
Answer: $c = 2$.

Solved chapter practice

Practice. Evaluate $\lim_{x\to 0} \dfrac{\sqrt{1+3x}-1}{x}$.

Solution. Rationalize: $\frac{\sqrt{1+3x}-1}{x}\,\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} = \frac{3}{\sqrt{1+3x}+1}$. Now substitution is legal, so the limit is $3/2$.

Mastery practice A. Reconstruct limits: what the function is approaching from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a limits: what the function is approaching problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 3 The derivative: a local rate

Rebuild the chapter from first principles

The claim I need to own is this: The model in The derivative: a local rate controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of the derivative: a local rate. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach the derivative: a local rate on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from limits: what the function is approaching to the chain rule and implicit structure. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

The derivative stopped feeling mysterious when I read the difference quotient as units: output change divided by input change. The limit does not invent a slope; it asks what the average slopes settle toward as the interval shrinks.

The easy version

Zoom in on a smooth curve until it looks like a line. The slope of that tiny line is the derivative.

Derivative at a point.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The quotient is an average rate over a small interval. The limit collapses the interval and leaves the instantaneous rate. Geometrically it is the tangent slope; physically it turns position into velocity.

Worked example

Find the derivative of $f(x) = x^2$ from the definition.
Solution.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

The cancellation is the seam: before expanding, the expression is $0/0$; after exposing the common h, the local rate becomes visible.

Differentiability implies continuity. The reverse fails: $|x|$ is continuous at 0 but has left slope -1 and right slope 1.

Basic rules

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad (cf+g)' = cf' + g',$$
$$(fg)' = f'g + fg', \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Sanity check

A derivative should have the right scale. If $f(x) = x^5$, then $f'(x)$ should behave like x^4, not x^6. If position is measured in kilometers and time in hours, velocity cannot still be labeled kilometers.

Checkpoint

Differentiate $q(x) = \frac{x^2+1}{x-1}$.
Answer: $q'(x) = \frac{x^2-2x-1}{(x-1)^2}$.

Solved chapter practice

Practice. Use the definition to find $f'(x)$ for $f(x) = 1/x$.
Solution. $\frac{f(x+h)-f(x)}{h} = \frac{1/(x+h)-1/x}{h} = \frac{-h}{x(x+h)h} = -\frac{1}{x(x+h)}$. Letting $h \to 0$ gives $f'(x) = -1/x^2$ for $x \neq 0$.

Mastery practice A. Reconstruct the derivative: a local rate from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a the derivative: a local rate problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 4 The chain rule and implicit structure

How I actually think about it

The chain rule is bookkeeping for dependency. I write the layers before differentiating because that keeps me from losing an inner derivative. Implicit differentiation is the same dependency story, except y has been left unnamed as a function of x.

The easy version

If one machine feeds another machine, multiply how fast the outside reacts by how fast the inside changes.

The idea

The chain rule is the derivative of a pipeline. If x changes, the inner function changes; that altered output becomes the input change seen by the outer function.

Rebuild the chapter from first principles

The claim I need to own is this: The chain rule is the derivative of a pipeline. If x changes, the inner function changes; that altered output becomes the input change seen by the outer function. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of the chain rule and implicit structure. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach the chain rule and implicit structure on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the derivative: a local rate to exponential, logarithmic, and trigonometric rates. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

$$\boxed{\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).}$$

The move

Name the layers before differentiating. For $\sqrt{1+e^{3x}}$, the layers are square root, 1+, exponential, and $3x$. Work from outside inward and multiply by each inner rate.

$$\frac{d}{dx}\sqrt{1+e^{3x}} = \frac{1}{2\sqrt{1+e^{3x}}} \cdot e^{3x} \cdot 3.$$

Implicit differentiation is the same rule when y is defined by a relation. Differentiate both sides with respect to x and attach y' whenever a y is differentiated.

Worked example

For $x^2 + xy + y^2 = 7$, find dy/dx.
Solution.

$$2x + (xy' + y) + 2yy' = 0,$$

so

$$y'(x+2y) = -(2x+y), \qquad \boxed{y' = -\frac{2x+y}{x+2y}}.$$

Inverse functions

If f is differentiable and one-to-one near a, with $f'(a) \neq 0$, then

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

This is a reciprocal of rates at corresponding points, not at the same input unless the inputs happen to match.

Checkpoint

If $f(2) = 5$ and $f'(2) = 7$, find $(f^{-1})'(5)$.
Answer: $1/7$.

Solved chapter practice

Practice. If $x^2 + xy + y^2 = 12$, find dy/dx and the slope at $(2, 2)$.
Solution. Differentiate: $2x + xy' + y + 2yy' = 0$. Hence $y'(x + 2y) = -(2x + y)$, so $y' = -(2x + y)/(x + 2y)$. At $(2, 2)$ the slope is $-6/6 = -1$.

Mastery practice A. Reconstruct the chain rule and implicit structure from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a the chain rule and implicit structure problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 5 Exponential, logarithmic, and trigonometric rates

Rebuild the chapter from first principles

The claim I need to own is this: The model in Exponential, logarithmic, and trigonometric rates controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of exponential, logarithmic, and trigonometric rates. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach exponential, logarithmic, and trigonometric rates on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the chain rule and implicit structure to linearization: the best local model. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

These derivatives are easier to remember by behavior than by a table. Exponentials reproduce themselves, logarithms turn multiplication into addition, and trigonometric derivatives cycle. The formulas become less fragile when I know what kind of motion each function represents.

The easy version

Exponential growth keeps its shape, logs undo exponentials, and sine and cosine pass the derivative back and forth.

The core derivatives are

$$(e^x)' = e^x, \quad (a^x)' = a^x \ln a, \quad (\ln x)' = \frac{1}{x},$$
$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x.$$

Logarithmic differentiation is useful when products, quotients, and variable exponents are tangled together.

Worked example

Differentiate $y = x^x$ for $x > 0$.
Solution. Take logs: $\ln y = x \ln x$. Then

$$\frac{y'}{y} = \ln x + 1,$$

so $\boxed{y' = x^x(\ln x + 1)}$.

Common miss

$\ln(a + b)$ does not split. Log rules turn products into sums and powers into coefficients; they do nothing comparable for addition.

Solved chapter practice

Practice. Differentiate $y = (\sin x)^x$ on an interval where $\sin x > 0$.
Solution. Take logs: $\ln y = x \ln(\sin x)$. Differentiate to get $y'/y = \ln(\sin x) + x \cot x$. Therefore $y' = (\sin x)^x[\ln(\sin x) + x \cot x]$.

Mastery practice A. Reconstruct exponential, logarithmic, and trigonometric rates from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a exponential, logarithmic, and trigonometric rates problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 6 Linearization: the best local model

Rebuild the chapter from first principles

The claim I need to own is this: The model in Linearization: the best local model controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of linearization: the best local model. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach linearization: the best local model on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from exponential, logarithmic, and trigonometric rates to what derivatives reveal globally. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

Linearization is not a trick for avoiding a calculator; it is the first-order model hiding inside every differentiable function. I always write the center because the approximation is only honest near that point.

The easy version

Near a point, replace a smooth curve with the tangent line touching it there.

Near $x = a$, a differentiable function behaves like its tangent line:

$$\boxed{f(x) \approx L(x) = f(a) + f'(a)(x - a).}$$

Worked example

Approximate $\sqrt{4.1}$.
Solution. Use $f(x) = \sqrt{x}$ near $a = 4$:

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

Thus $\sqrt{4.1} \approx 2 + 0.025 = 2.025$. The true value is slightly smaller because $\sqrt{x}$ is concave down.

Differentials package the same idea: $dy = f'(x)\,dx$. They are especially useful for propagated error. Relative error is roughly $|dy/y|$; percentage error is $100|dy/y|\%$.

Sanity check

Approximation is local. I always state the center a and ask whether the target is actually close to it. A beautiful tangent line used far away is still a bad model.

Solved chapter practice

Practice. Use linearization at $a = 8$ to approximate $\sqrt[3]{8.12}$.
Solution. For $f(x) = x^{1/3}$, $f(8) = 2$ and $f'(8) = 1/(3 \cdot 8^{2/3}) = 1/12$. Thus $L(x) = 2 + (x - 8)/12$, so $\sqrt[3]{8.12} \approx 2 + 0.12/12 = 2.01$.

Mastery practice A. Reconstruct linearization: the best local model from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a linearization: the best local model problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 7 What derivatives reveal globally

Rebuild the chapter from first principles

The claim I need to own is this: The model in What derivatives reveal globally controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of what derivatives reveal globally. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach what derivatives reveal globally on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from linearization: the best local model to optimization and related rates. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

A derivative is local data, but its sign across an interval tells a global story. I do not classify a point merely because $f' = 0$ or $f'' = 0$; I check what changes on either side.

The easy version

Make a sign chart for the slope. Positive means climbing, negative means falling, and a sign change identifies the turn.

Critical number. A point c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.

Critical numbers are candidates for local extrema, not automatic winners. On a closed interval, the absolute extrema occur among critical numbers and endpoints.

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), some $c \in (a, b)$ satisfies

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The theorem connects one local rate to the average rate over the entire interval.

First and second derivative tests

- $f' > 0$: increasing; $f' < 0$: decreasing.
- $f'' > 0$: concave up; $f'' < 0$: concave down.
- A sign change in f' from $+$ to $-$ gives a local maximum.
- An inflection point requires a change in concavity, not merely $f'' = 0$.

Checkpoint

For $f(x) = x^3 - 3x$, critical numbers are $x = \pm 1$. Classify them.
Answer: local maximum at $x = -1$ and local minimum at $x = 1$.

Solved chapter practice

Practice. Find and classify the critical points of $f(x) = x^4 - 4x^2$.
Solution. $f'(x) = 4x(x^2 - 2)$, so $x = 0, \pm\sqrt{2}$. Since $f''(x) = 12x^2 - 8$, $f''(0) = -8 < 0$, giving a local maximum at 0. At $\pm\sqrt{2}$, $f'' = 16 > 0$, giving local minima.

Mastery practice A. Reconstruct what derivatives reveal globally from a blank page. Include

the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a what derivatives reveal globally problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 8 Optimization and related rates

Rebuild the chapter from first principles

The claim I need to own is this: The model in Optimization and related rates controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of optimization and related rates. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.

- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach optimization and related rates on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from what derivatives reveal globally to the integral: accumulation with structure. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

The hard part is almost never taking the derivative. It is choosing variables and translating the constraint before doing calculus. I write units beside rates because related-rates errors often look algebraically clean but physically impossible.

The easy version

Turn the story into one equation, reduce it to one changing variable, and only then differentiate.

The move

For optimization: draw the object, name variables with units, write the target, write the constraint, reduce to one variable, find all candidates, and compare. The calculus is usually the shortest part.

Worked example

A rectangle has perimeter 20. Maximize its area.
Solution. Let sides be x and y. The constraint $2x + 2y = 20$ gives $y = 10 - x$. Then

$$A(x) = x(10 - x), \qquad A'(x) = 10 - 2x.$$

The critical point is $x = 5$, so $y = 5$ and the maximum area is 25. Endpoints give area 0, confirming the interior maximum.

Related-rates problems differentiate a constraint with respect to time before substituting the instant's data.

Worked example

A circle's radius grows at 2 cm/s. How fast does area grow when $r = 3$ cm?
Solution. From $A = \pi r^2$,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(3)(2) = 12\pi \text{ cm}^2/\text{s}.$$

Solved chapter practice

Practice. A closed cylinder has volume 54π. Find the radius and height minimizing surface area.
Solution. $\pi r^2 h = 54\pi$ gives $h = 54/r^2$. Surface area is $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 108\pi/r$. Then $S' = 4\pi r - 108\pi/r^2 = 0$, so $4r^3 = 108$ and $r = 3$. Thus $h = 54/9 = 6$.

Mastery practice A. Reconstruct optimization and related rates from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.
Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a optimization and related rates problem but never states a model or checks the result. Write the shortest useful review of that solution.
Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 9 The integral: accumulation with structure

Rebuild the chapter from first principles

The claim I need to own is this: The model in The integral: accumulation with structure controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of the integral: accumulation with structure. List the known quantities, the unknown, the units or types, and any hidden constraint.

2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.

3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that

I can audit it later.

4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach the integral: accumulation with structure on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from optimization and related rates to cumulative practice. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

How I actually think about it

I think of an integral as net change assembled from tiny contributions, not automatically as geometric area. The Fundamental Theorem matters because it says accumulation and instantaneous rate are inverse views of the same process.

The easy version

Add many tiny signed pieces; an antiderivative lets us compute that total from the endpoints.

A Riemann sum approximates accumulated change:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

The definite integral is the limit of these sums as the partition gets arbitrarily fine:

$$\int_a^b f(x)\,dx.$$

It is signed accumulation. Area below the axis counts negatively.

Fundamental Theorem of Calculus

If f is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

and if $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

One part differentiates accumulated area; the other evaluates it through any antiderivative.

Worked example

Differentiate $G(x) = \int_1^{x^2} \cos(t^3)\,dt$.
Solution. The outer operation is accumulated area with upper limit x^2. By FTC plus chain rule,

$$G'(x) = \cos((x^2)^3)\cdot 2x = 2x\cos(x^6).$$

No antiderivative of $\cos(t^3)$ is needed.

Substitution reverses the chain rule:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \quad u = g(x).$$

Checkpoint

Evaluate $\int_0^1 2xe^{x^2}\,dx$.
Answer: $e - 1$.

Solved chapter practice

Practice. Differentiate $F(x) = \int_{x^2}^{3x} e^{t^2}\, dt$.

Solution. Use the variable-endpoint rule: upper contribution minus lower contribution. Thus $F'(x) = e^{(3x)^2}(3) - e^{(x^2)^2}(2x) = 3e^{9x^2} - 2xe^{x^4}$.

Mastery practice A. Reconstruct the integral: accumulation with structure from a blank page. Include the central definition, the governing rule, one required assumption, and one boundary case.

Solution. Start from the blue idea box, but rewrite it in my own words. Identify the objects and state, name the rule before using it, copy the relevant assumption from the details audit, and choose a boundary where that assumption becomes equality or fails. A complete solution explains why each part belongs; a list of vocabulary is not enough.

Mastery practice B. A classmate gets a polished-looking answer to a the integral: accumulation with structure problem but never states a model or checks the result. Write the shortest useful review of that solution.

Solution. I would ask four concrete questions: What exactly are the inputs and desired output? Which rule licenses the main step? Which assumption could invalidate it? Which independent check supports the conclusion? If the answer cannot survive those four questions, I treat the numerical or verbal result as unverified.

CHAPTER 10 Cumulative practice

Rebuild the chapter from first principles

The claim I need to own is this: The model in Cumulative practice controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of cumulative practice. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach cumulative practice on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from the integral: accumulation with structure to written solutions. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Find the domain of $f(x) = \dfrac{\sqrt{2-x}}{x^2-1}$.
2. Evaluate $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$.
3. Find constants a, b so $f(x) = \begin{cases} ax+b, & x<2 \\ x^2, & x \geq 2 \end{cases}$ is differentiable at 2.
4. Differentiate $y = \dfrac{e^{x^2}\sin x}{1+x^3}$.
5. For $x^2 + 4y^2 = 20$, find the tangent line at $(2, 2)$.
6. Use linearization to approximate $(1.02)^5$.
7. Prove that $x^3 + x - 1 = 0$ has exactly one real root.
8. Find the absolute extrema of $f(x) = x + 4/x$ on $[1, 5]$.
9. A 10-ft ladder slides down a wall. The bottom moves away at 3 ft/s. Find the top's velocity when the bottom is 6 ft from the wall.
10. A box with square base and no top must hold volume 32. Find dimensions minimizing surface area.

11. Differentiate $H(x) = \int_{x}^{x^2} \sqrt{1+t^4}\,dt$.

12. Evaluate $\int_0^2 x\sqrt{x^2+1}\,dx$.

CHAPTER 11 Written solutions

Rebuild the chapter from first principles

The claim I need to own is this: The model in Written solutions controls the work. I do not count that as learned just because the sentence looks familiar. I should be able to name the objects, state what changes and what stays fixed, recover the rule from the definitions, and explain why the result is allowed.

The move

My working sequence

1. **Translate.** Rewrite the prompt in the language of written solutions. List the known quantities, the unknown, the units or types, and any hidden constraint.
2. **Choose.** State the theorem, model, algorithm, accounting rule, or physical law before substituting values. This is where I make the assumption visible.
3. **Execute.** Work one justified step at a time and keep intermediate state readable enough that I can audit it later.
4. **Verify.** Check a boundary case, sign, unit, invariant, order of magnitude, or alternative derivation. Then translate the result back into an ordinary sentence.

Details I would refuse to skip

- **Definitions:** I should be able to define every technical noun in the chapter without using that same noun in the definition.
- **Assumptions:** I write the condition that makes the method legal beside the method itself. A correct procedure under the wrong assumptions is still a wrong answer.
- **Representation:** I move freely among words, equations or code, a diagram, and a small numerical example. If one representation disagrees with another, that disagreement is useful evidence.
- **Failure modes:** I keep the nearest counterexample in mind: the boundary where the rule changes, the invariant breaks, or the model stops matching reality.
- **Communication:** My final answer names what the result means and what it does not establish. I do not let notation do the explanatory work for me.

Sanity check

Closed-note check. Can I teach written solutions on a blank page, derive or reconstruct its main rule, work a fresh example, and diagnose a deliberately broken solution? If I can only recognize the finished work, I am not done.

How this chapter connects

I read this chapter as the bridge from cumulative practice to the cumulative course model. The earlier material supplies the language and constraints; this chapter adds a new reasoning move; the later material asks me to use that move without being told its name. That connection matters because exams and real projects mix chapters on purpose.

1. Domain

We need $2 - x \geq 0$, so $x \leq 2$, and $x^2 - 1 \neq 0$, so $x \neq \pm 1$. Thus

$$(-\infty, -1) \cup (-1, 1) \cup (1, 2].$$

2. Rationalized limit

Multiply by the conjugate:

$$\frac{\sqrt{1+x}-1}{x}\,\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1+x}+1} \to \frac{1}{2}.$$

3. Differentiable join

Continuity at 2 requires $2a + b = 4$. Matching derivatives requires $a = 2(2) = 4$. Hence $b = -4$.

4. Product and quotient structure

Let $N = e^{x^2} \sin x$. Then $N' = e^{x^2}(2x \sin x + \cos x)$. Therefore

$$y' = \frac{e^{x^2}(2x \sin x + \cos x)(1 + x^3) - 3x^2 e^{x^2} \sin x}{(1 + x^3)^2}.$$

5. Implicit tangent

Differentiate: $2x + 8yy' = 0$, so $y' = -x/(4y)$. At $(2, 2)$ the slope is $-1/4$. The line is

$$y - 2 = -\frac{1}{4}(x - 2).$$

6. Linearization

For $f(x) = x^5$ at $a = 1$, $f(1) = 1$ and $f'(1) = 5$. Thus

$$f(1.02) \approx 1 + 5(0.02) = 1.10.$$

7. Exactly one root

Let $g(x) = x^3 + x - 1$. Since $g(0) = -1$ and $g(1) = 1$, continuity gives at least one root in $(0, 1)$. Also $g'(x) = 3x^2 + 1 > 0$ for every real x, so g is strictly increasing and cannot have two roots. Exactly one exists.

8. Absolute extrema

$f'(x) = 1 - 4/x^2$, so the interior critical point is $x = 2$. Compare

$$f(1) = 5, \qquad f(2) = 4, \qquad f(5) = \frac{29}{5}.$$

The absolute minimum is 4 at $x = 2$; the absolute maximum is $29/5$ at $x = 5$.

9. Sliding ladder

Let x be bottom distance and y be height. Then $x^2 + y^2 = 100$. Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

When $x = 6$, $y = 8$. Hence

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{8}(3) = -\frac{9}{4} \text{ ft/s}.$$

The negative sign means downward.

10. Open box

Let base side be x and height h. Volume gives $x^2h = 32$, so $h = 32/x^2$. Surface area is

$$S(x) = x^2 + 4xh = x^2 + \frac{128}{x}, \qquad x > 0.$$

Then $S'(x) = 2x - 128/x^2$. Setting this to zero gives $2x^3 = 128$, so $x = 4$ and $h = 2$. Since $S''(x) = 2 + 256/x^3 > 0$, this is the minimum.

11. Variable endpoints

Using FTC on each endpoint,

$$H'(x) = \sqrt{1 + (x^2)^4}(2x) - \sqrt{1 + x^4} = 2x\sqrt{1 + x^8} - \sqrt{1 + x^4}.$$

12. Substitution

Let $u = x^2 + 1$, so $du = 2x\,dx$. Bounds change from $x = 0, 2$ to $u = 1, 5$:

$$\int_0^2 x\sqrt{x^2 + 1}\,dx = \frac{1}{2}\int_1^5 u^{1/2}\,du = \frac{1}{3}\left(5^{3/2} - 1\right).$$

Closing frame

The durable thread in 31A is local-to-global reasoning. A limit defines a local behavior, a derivative turns it into a local rate, the Mean Value Theorem carries that rate across an interval, and the integral rebuilds global accumulation from local pieces. If those transitions feel natural, the course is doing its job.